

# Interplay between quasi-periodicity and disorder in quantum spin chains in a magnetic field

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We study the interplay between disorder and a quasi periodic coupling array in an external magnetic field in a spin- $\frac{1}{2}$   $XXZ$  chain. A simple real space decimation argument is used to estimate the magnetization values where plateaux show up. The latter are in good agreement with exact diagonalization results on fairly long  $XX$  chains. Spontaneous susceptibility properties are also studied, finding a logarithmic behaviour similar to the homogeneously disordered case.

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Since their discovery in 1984 [1], the properties of quasi-crystals have been a source of sustained interest. Many theoretical efforts on Ising models in Penrose lattices [2] and  $XY$  Fibonacci spin chains [3–5] have revealed interesting magnetic orderings associated to the quasi-periodicity of these structures. Such kind of spin arrays have been found in recently synthesized rare earth ( $\mathcal{R}$ )  $ZnMg-\mathcal{R}$  quasi-crystals (see *e.g.* [6]) whose  $\mathcal{R}$  elements have well localized  $4f$  magnetic moments. The study of quasiperiodic 1D chains has recently received renewed attention [7,8] and interesting properties have been elucidated. In [7] a system of spinless fermions in a quasiperiodic lattice potential was studied within perturbation theory, where it was shown that its behaviour is different from both the periodic and the disordered cases: While in the case of a periodic potential one may have a metal-insulator transition only if the potential is commensurate, in the disordered case, the potential is relevant irrespective of the position of the Fermi level. In the quasiperiodic case, two different situations arise, depending on whether the Fermi level coincides with one of the main frequencies of the Fourier spectrum of the quasiperiodic potential or not. In the first case, the situation turns out to be similar to the periodic case while in the second, at a perturbative level, the metal-insulator transition point is strongly modified. These predictions have been also verified numerically in [8]. Motivated by these studies we have recently analyzed the effect of an external magnetic field in a quasiperiodic spin chain, and found that the magnetization curve has a very interesting nature that could be predicted using a decimation procedure and Abelian bosonization [9]. For the Fibonacci case, in particular, one can reproduce the main plateaux within the bosonization approach by approximating the quasiperiodic modulation by considering a subset of the main Fourier frequencies. From the experimental point of view, interest of quasiperiodic systems arise from artificially grown quasi-periodic heterostructures [13], quantum dot crystals [14] and magnetic multilayers [15].

In this short note we go one step further to analyze the interplay between a quasiperiodic array of couplings and disorder in a  $XXZ$  spin chain in the presence of an external magnetic field. Using a simple decimation procedure

we predict the appearance of plateaux in the magnetization curve at values of  $M$  which depend on both the quasiperiodicity and the strength of the disorder. As in the case of  $p$ -merized chains the presence of binary disorder results in a shift on the plateaux positions as a function its strength, while for the Gaussian case the plateaux are wiped out. We have also studied the effects of a binary disorder on the double frequency  $XX$  chain studied in [9], where it turns out that the plateaux structure of the pure case disappears. We have checked this behaviour by studying numerically fairly long  $XX$  systems. We also predict a logarithmic behaviour of the susceptibility at low fields by extending the arguments in [16] and have also verified this behaviour numerically. Other authors have also studied random spin systems recently [10]–[12].

Let us consider the antiferromagnetic system

$$H_{qp} = J \sum_n (1 + \epsilon_n) (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z) - h \sum_n S_n^z, \quad (1)$$

where  $S^x, S^y, S^z$  denote the standard spin- $\frac{1}{2}$  matrices, in a magnetic field  $h$  applied along the anisotropy direction ( $|\Delta| \leq 1$ ). Here, the coupling modulation is introduced via the  $\epsilon_n$  parameters defined as  $\epsilon_n = \sum_\nu \delta_\nu \cos(2\pi \omega_\nu n)$ , so quasi-periodicity arises upon choosing an irrational subset of frequencies  $\omega_\nu$  with amplitudes  $\delta_\nu$ .

Furthermore, the couplings  $J_n$  are randomly distributed. Specifically, we consider a binary distribution of strength  $p$  ( $p = 0$  corresponds to the pure quasiperiodic case while  $p = 1$  corresponds to the uniform chain),

$$P(J_i) = p \delta(J_i - U) + (1 - p) \delta(J_i - J(1 + \epsilon_n)), \quad (2)$$

with  $\epsilon_n$  defined as above, along with a Gaussian disorder  $P(J_i) \propto \exp -\frac{(J_i - \bar{J}_i)^2}{2\sigma_i^2}$ . These distributions, taken with same mean and variance, are built to enforce quasiperiodicity. Thus, on average  $\epsilon_n$  is a measure of the couplings quasiperiodicity. In what follows we assume that  $U$  is the smallest coupling

We will follow the decimation procedure as described in [16] to obtain the value of the magnetization for the main plateaux. In our problem (which is at  $T = 0$ ) the energy scale is provided by the magnetic field, and in order to compute the magnetization, decimation has to be stopped at an energy scale of the order of the magnetic field. We assume that all spins coupled by bonds stronger than the magnetic field form singlets and do not contribute to the magnetization, whereas spins coupled by weaker bonds are completely polarized. The magnetization is thus proportional to the fraction of remaining spins at the step where we stop decimation. This simple argument happens to apply well to the binary distribution, provided the energy scales of the involved exchanges are well separated.

In studying irrational frequencies or other quasiperiodic modulations, it is natural to analyze the case of the Fibonacci chain, a coupling array  $J_A = J(1 + \delta)$ ,  $J_B = J(1 - \delta)$  generated by iterating the substitution rules  $B \rightarrow A$  and  $A \rightarrow AB$  [5], [7], [9], [17]; with the distribution (2)  $P(J_i) = p \delta(J_i - U) + (1 - p) \delta(J_i - J_{A,B})$ .

We evaluate by decimation the magnetization of the widest plateaux in the *strong* coupling limit ( $\delta \rightarrow \pm 1$ ). There are two different cases to consider, according to  $\delta \approx -1$ , i.e.,  $J_B \gg J_A$ , and the opposite situation for  $\delta \approx 1$ .

Starting from saturation, in the first case the magnetic field is lowered until it reaches the value  $h_c \approx J_B$  at which the type- $B$  bonds experience a transition from the state of maximum polarization to the singlet state. The magnetization at this plateau is then obtained by decimating the  $B$  bonds. This yields:

$$\langle M \rangle = 1 - 2 \frac{N_B}{N_T} = 1 - 2(1 - p) \frac{1}{\gamma^2}, \quad (3)$$

where  $N_T = N_U + N_A + N_B$  denotes the total number of bonds,  $N_{A,B}$  the number of  $A$  and  $B$  bonds respectively and  $\gamma^2 = (N_A + N_B)/N_B$ . For a large iteration number of the rules referred to above ( $N_T \rightarrow \infty$ ),  $N_A/N_B$  approaches the golden mean  $\gamma = \left(\frac{1+\sqrt{5}}{2}\right)$ . In the  $p = 0$  limit, we recover the results in [9] and a non-vanishing  $p$  results in a shift of the position of the plateau.

In the second case,  $J_A \gg J_B$ , we have to distinguish two different unit cells since type- $A$  bonds can appear either in pairs (forming trimers) or isolated (forming dimers). It can be readily checked that when lowering the magnetic field from saturation the first spins to be decimated correspond to those forming trimers. We then have a plateau (the nearest to saturation) at:

$$\langle M \rangle_1 = 1 - 2 \frac{N_{AA}}{N_T} = 1 - 2(1 - p)^2 \frac{1}{\gamma^3}, \quad (4)$$

where  $N_{AA}$  refers to  $A$  pairs. The second plateau is obtained after decimation the type- $A$  bonds, and then we must consider all the sequences  $J_A$  between the others bonds. That gives for the second plateau:

$$\begin{aligned} \langle M \rangle_2 = & \langle M_1 \rangle - 2 \left[ (1 - p)^3 \frac{1}{\gamma^4} + \right. \\ & \left. 2p(1 - p)^2 \frac{1}{\gamma^2} + p^2(1 - p) \frac{1}{\gamma} \right]. \end{aligned} \quad (5)$$

Again, we recover our results in [9] for  $p = 0$ .

With this simple technique, one can predict the presence and position of the plateaux, provided that there is a finite difference between the highest values of the couplings in the inequivalent sites.

Since the decimation procedure applies for generic XXZ chains [16], [18], we conclude that the emergence of these strong coupling plateaux is a generic feature, at least with an antiferromagnetic anisotropy parameter  $0 < \Delta < 1$ .

To enable an independent check of these assertions, we turn to a numerical diagonalization of the Hamiltonian (1) contenting ourselves with the analysis of the particular case  $\Delta = 0$ . This allows us to explore rather long chains using a fair number of disorder realizations (whose magnetization properties on the other hand, are self-averaging). In Fig (1) we show respectively the whole magnetization curves obtained for various disorder concentrations  $p = 0, 0.2, 0.4, 0.6, 0.8$  and 1 averaging on  $5 \times 10^4$  samples of  $L = f(18) = 2584$  sites under the exchange disorder (2), with  $\delta = 0.95$  and  $U = 0.2$ . It can be readily verified that a set of robust plateaux emerges quite precisely at the critical magnetizations given by Eq. (4) for the plateau closest to saturation and by Eq. (5) for the second one.

In a previous work [9], we have observed that the magnetization curve for the Fibonacci chain, could be well approximated by considering a rather small subset of the main frequencies in its Fourier spectrum. Here we study a two-frequency case, for  $\omega_1 = 5/8$  and  $\omega_2 = 7/8$ , in the presence of disorder, where it is observed that the plateaux are erased even by small disorder (see Fig. (2)). It is interesting to observe that in contrast to the Fibonacci situation studied above, in which the plateaux structure is robust and just shifts with the strength of the disorder, here the plateaux seem to smear out even for a small value of  $p$ .

It is important to stress that the derivation of our results for the quantization conditions Eqs. (3)-(5) rely strongly on the discreteness of the probability distribution and would not be applicable to an arbitrary continuous exchange disorder. In fact, for the Gaussian case referred to above it turns out that no traces of plateaux can be observed. This is corroborated in Fig. (3), where we see that the plateaux structure is smoothed when the standard deviation is increased. Here the sampling was increased up to  $4 \times 10^4$  realizations though the length of the chain was reduced to  $L = f(18)$ , as the CPU time per spectrum grows as  $L^2$ . However, preliminary computations using larger chains yielded no substantial differences.

For homogeneously disordered chains, one can use the decimation procedure of [16] along with the universality of the fixed point, to show that either for discrete or continuous distributions the low field magnetic susceptibility behaves according to

$$\chi_z \propto \frac{1}{H[\ln(H^2)]^3}. \quad (6)$$

Following a simple argument based on random walk motion used in [19], it can be readily shown that for  $\Delta = 0$  (or  $XX$  chains), these arguments can be extended to the case of a disordered Fibonacci chain. It is interesting to note that the effect of the disorder is crucial since it changes the power law behaviour of the free Fibonacci chain obtained in [4] to a logarithmic one. This can be clearly observed in the insets of Figs. (4), (5) where the validity of these arguments seem to apply over more than two decades.

To summarize, we have studied the effect of disorder on the plateaux structure in quasiperiodic  $XXZ$  chains under an external magnetic field. By means of a simple real space decimation procedure we found the values of the magnetization for which the main plateaux emerge, Eqs. (3)-(5). This was tested by numerical diagonalizations of large  $XX$  chains finding a remarkable agreement with the quantization conditions in a variety of scenarios. Since the decimation scheme applies for generic  $XXZ$  chains [18], we conclude that the appearance of these plateaux is a generic feature, at least with an antiferromagnetic anisotropy parameter  $0 < \Delta < 1$ . This issue still awaits numerical confirmation on sufficiently long chains using state of the art methodologies such as density matrix renormalization group [20]. Finally, we have also studied the low magnetic field susceptibility which exhibits a clear logarithmic behavior, Eq. (6). We trust this work will convey a motivation for both experimental and numerical studies.

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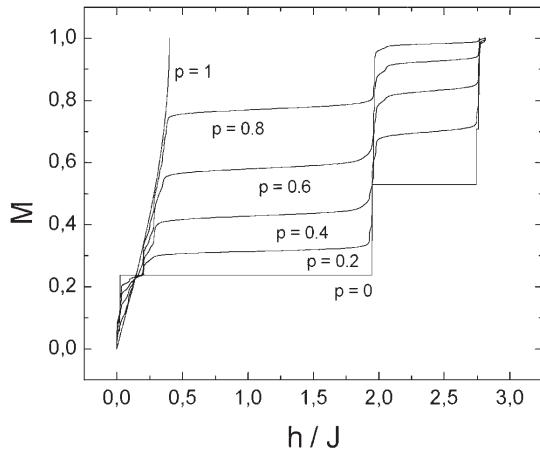


FIG. 1. Magnetization curves of modulated XX Fibonacci spin chains, immersed in disordered binary backgrounds of strength  $p$  after averaging over  $5 \times 10^4$  samples with  $f(18) = 2584$  sites,  $\delta = 0.95$ ,  $U = 0.2$  and  $p = 0, 0.2, 0.4, 0.6, 0.8, 1$  in ascending order. The left and rightmost lines denote respectively the pure uniform and pure Fibonacci cases.

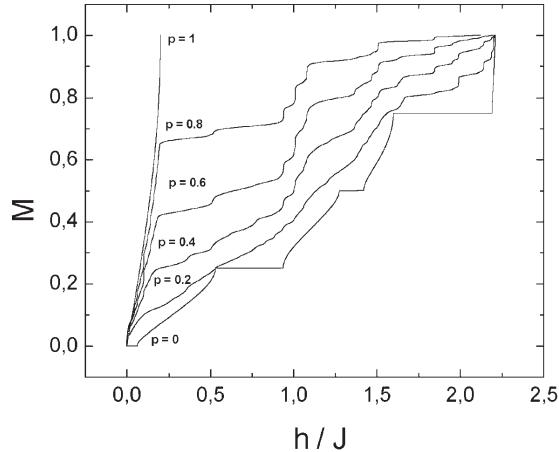


FIG. 2. Double frequency magnetization curves of the XX chain for  $\omega_1 = 5/8$  and  $\omega_2 = 7/8$  with amplitudes  $\delta_1 = 0.2$  and  $\delta_2 = 0.3$ , with  $U = 0.1$  and  $10^4$  spins over 100 samples, immersed in disordered binary backgrounds of strength  $p = 0, 0.2, 0.4, 0.6, 0.8$  and 1 in ascending order

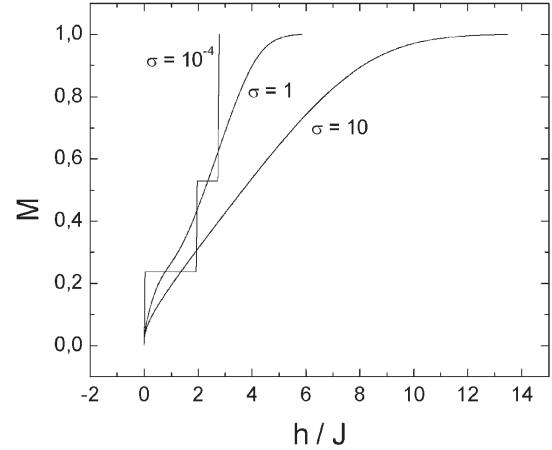


FIG. 3. Magnetization curves of modulated XX Fibonacci spin chains, immersed in Gaussian exchange distributions, after averaging over  $4 \times 10^4$  samples with  $f(18) = 2584$  sites,  $\delta = 0.95$  and increasing standard deviation from left to right (note that the leftmost is practically the pure Fibonacci case).

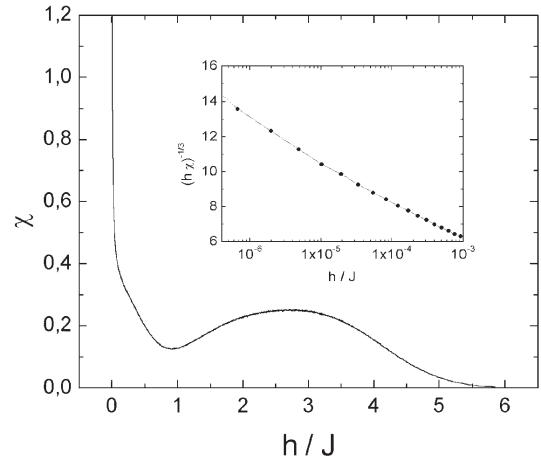


FIG. 4. Magnetic susceptibility of modulated XX Fibonacci spin chains, immersed in a Gaussian exchange distribution ( $\sigma = 1$ ), after averaging over  $4 \times 10^4$  samples with  $f(18) = 2584$  sites,  $\delta = 0.95$ . The inset show the susceptibility behavior at low magnetic fields which follows closely the logarithmic regime predicted in the text.

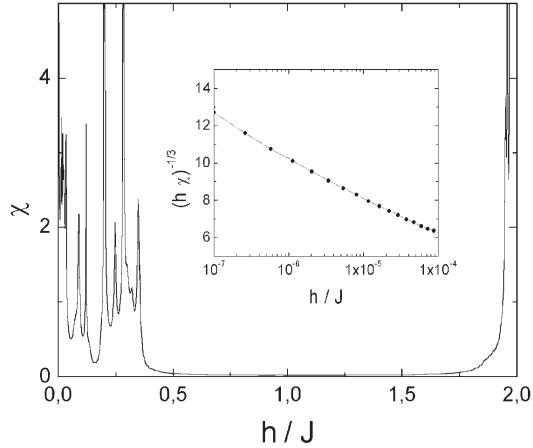


FIG. 5. Magnetic susceptibility of modulated XX Fibonacci spin chains, immersed in a binary exchange disorder of strength  $p = 0.6$  averaged over  $5 \times 10^4$  samples with  $f(18) = 2584$  sites,  $\delta = 0.95$ . The inset show the susceptibility behavior at low magnetic fields that like the Gaussian disorder follows closely the logarithmic regime.